NASA TECHNICAL TRANSLATION

NASA TT F-261

NASA TT F-261

THE PROBLEM OF THE CONVERGENCE OF THE METHOD OF INTEGRAL RATIOS

by V. V. Bobkov

From *Doklady Akademii Nauk BSSR*,
Vol. 8, No. 1, 1964.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • SEPTEMBER 1965

NASA TT F-261

THE PROBLEM OF THE CONVERGENCE OF THE METHOD OF INTEGRAL RATIOS

By V. V. Bobkov

Translation of "K voprosu o skhodimosti metoda integral'nykh sootnosheniy." Doklady Akademii Nauk BSSR, Vol. 8, No. 1, pp. 5-9, 1964.

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

For sale by the Clearinghouse for Federal Scientific and Technical Information
Springfield, Virginia 22151 – Price $1.00

THE PROBLEM OF THE CONVERGENCE OF THE METHOD OF INTEGRAL RATIOS

V. V. Bobkov

The method proposed by A. A. Dorodnitsyn for the approximate solution of differential equations in partial derivatives (Ref. 1) has been successfully applied to the solution of a wide variety of problems (Ref. 2); however, the convergence of this method, generally speaking, has still not been demonstrated.

The method of integral ratios is applicable to various types of problems. Ordinarily, when it is used the initial differential equations are written in divergent form (Ref. 1).

We will consider differential equations of the hyperbolic type, writing them for this purpose in canonical form.

Let us take the following Cauchy problem:

$$u_{xy} = a(x, y)\,u_x + b(x, y)\,u_y + c(x, y)\,u + f(x, y), \tag{1}$$

$$u[x, L(x)] = \varphi(x), \quad \frac{\partial u[x, L(x)]}{\partial l} = \psi(x), \quad a_1 \leqslant x \leqslant a_2, \tag{2}$$

where l is the direction of a line nontangent to the curve $y = L(x)$ which nowhere takes a characteristic direction of curvature (i.e., which is piecewise smooth).

After the domain D, which is bounded by the curve $y = L(x)$ (for determinancy we will set $L(a_1) > (La_2))$ and by the characteristics $x = a_2$, $y = L(a_1)$, has been divided into N strips by the straight lines $y = y_n = L(a_2) + nh$, $h = d/N$, $d = L(a_1) - L(a_2)$, $n = 1, 2, \ldots, N - 1$, we integrate (1) over each of the strips:

$$\begin{gathered} u'(x, y_{n+1}) - u'(x, l_n) + l_n'(x)\,u_y(x, l_n) - \\ - b(x, y_{n+1})\,u(x, y_{n+1}) + b(x, l_n)\,u(x, l_n) = \\ = \int\limits_{l_n}^{y_{n+1}} [a(x, y)\,u_x + c(x, y)\,u - b_y(x, y)\,u + f(x, y)]\,dy, \\ x_{n+1} \leqslant x \leqslant a_2, \; n = 0, 1, \ldots, N - 1. \end{gathered} \tag{3}$$

Here x_{n+1} is the abscissa of the point of intersection of the line $y = y_{n+1}$ with the curve $y = L(x)$, while $y = l_n = l_n(x)$ coincides with $y = L(x)$ for $x_{n+1} \leqslant x \leqslant x_n$ and with $y = y_n$ for $x_n \leqslant x \leqslant a_2$.

After interpolating the functions u and u_x linearly according to their values at the boundaries of the sectors and completing the integration in (3), we obtain a system of N equations for finding approximate solutions u_n to problems (1), (2) on the lines $y = y_n$ (n = 1, 2, ..., N): /6

$$\alpha_{n+1} u'_{n+1} - \beta_n u'_{l_n} = \xi_{n+1} u_{n+1} + \gamma_n u_{l_n} + f_{n+1}, \tag{4}$$

with the supplementary conditions

$$u_{n+1}(x_{n+1}) = \varphi(x_{n+1}), \quad \frac{\partial u\,[x, L(x)]}{\partial l} = \psi(x), \quad a_1 \leqslant x \leqslant a_2, \tag{5}$$

where

$$u_{n+1} = u_{n+1}(x), \quad u_{l_n} = u_{l_n}(x) = \begin{cases} \varphi(x), & x_{n+1} \leqslant x \leqslant x_n, \\ u_n, & x_n \leqslant x \leqslant a_2, \end{cases} \quad h_n(x) = y_{n+1} - l_n(x);$$

$$\alpha_{n+1} = \alpha_{n+1}(x) = 1 - \frac{1}{h_n(x)} \int\limits_{l_n}^{y_{n+1}} a(x, y)\,[y - l_n(x)]\,dy,$$

$$\beta_n = \beta_n(x) = 1 + \frac{1}{h_n(x)} \int\limits_{l_n}^{y_{n+1}} a(x, y)\,(y_{n+1} - y)\,dy;$$

$$\xi_{n+1} = \xi_{n+1}(x) = b(x, y_{n+1}) + \frac{1}{h_n(x)} \times$$

$$\times \int\limits_{l_n}^{y_{n+1}} [c(x, y) - b_y(x, y)]\,[y - l_n(x)]\,dy;$$

$$\gamma_n = \gamma_n(x) = -b(x, l_n) + \frac{1}{h_n(x)} \times$$

$$\times \int\limits_{l_n}^{y_{n+1}} [c(x, y) - b_y(x, y)]\,(y_{n+1} - y)\,dy;$$

$$f_{n+1} = f_{n+1}(x) = \int\limits_{l_n}^{y_{n+1}} f(x, y)\, dy - \beta_n l_n'(x)\, u_y(x, l_n),$$

$$x_{n+1} \leqslant x \leqslant a_2, \quad n = 0, 1, \ldots, N-1.$$

For the error $\gamma_{n+1} = \gamma_{n+1}(x) = u(x, y_{n+1}) - u_{n+1}(x)$ we obtain:

$$\alpha_{n+1}\gamma_{n+1}' - \beta_n \gamma_{l_n}' = \xi_{n+1}\gamma_{n+1} + \eta_n \gamma_{l_n} + r_{n+1}, \quad \gamma_{n+1}(x_{n+1}) = 0,$$
$$x_{n+1} \leqslant x \leqslant a_2, \quad n = 0, 1, \ldots, N-1, \tag{6}$$

where

$$r_{n+1} = r_{n+1}(x) = \frac{1}{2} \int\limits_{l_n}^{y_{n+1}} \left\{ [c(x, y) - b_y(x, y)] \frac{\partial^2 u(x, \tilde{y}_{n+1})}{\partial y^2} + \right.$$

$$\left. + a(x, y) \frac{\partial^3 u(x, \tilde{\tilde{y}}_{n+1})}{\partial x \partial y^2} \right\} [y - l_n(x)]\,(y - y_{n+1})\, dy;$$

$$y_n < \tilde{y}_{n+1}; \quad \tilde{\tilde{y}}_{n+1} < y_{n+1}; \quad \gamma_{l_n} = \gamma_{l_n}(x) = \begin{cases} 0, & x_{n+1} \leqslant x \leqslant x_n, \\ \gamma_n, & x_n \leqslant x \leqslant a_2. \end{cases}$$

Both problem (6) and problem (4), (5) obtained earlier are solved consecutively beginning with n = 0. /7

We will show that as $h \to 0$, $u_n(x)$ converges toward $u(x, y_n)$ and will give the estimated error $\gamma = \max\limits_{x_n \leqslant x \leqslant a_2} \max\limits_{1 \leqslant n \leqslant N} |\gamma_n(x)|$.

We will divide problem (6) into N consecutively solvable Cauchy problems in the following manner:

$$P_n \Gamma_n' = Q_n \Gamma_n + R_n, \quad \Gamma_n(x_{N-n}) = \Phi_n, \quad x_{N-n} \leqslant x \leqslant x_{N-n-1},$$
$$n = 0, 1, \ldots, N-1, \tag{7}$$

where

$$P_n = \left[\begin{array}{ccc|cc} \alpha_{N-n} & 0 & 0 & 0 & 0 \\ -\beta_{N-n} & \alpha_{N-n+1} & 0 & 0 & 0 \\ 0 & -\beta_{N-n+1} & \alpha_{N-n+2} & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & -\beta_{N-1} & \alpha_N \end{array}\right], \quad \Gamma_n = \begin{bmatrix} \gamma_{N-n} \\ \gamma_{N-n+1} \\ \gamma_{N-n+2} \\ \vdots \\ \gamma_N \end{bmatrix},$$

$$Q_n = \left[\begin{array}{cccc|cc} \xi_{N-n} & 0 & 0 & & 0 & 0 \\ \eta_{N-n} & \xi_{N-n+1} & 0 & & 0 & 0 \\ 0 & \eta_{N-n+1} & \xi_{N-n+2} & & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & & \eta_{N-1} & \xi_N \end{array}\right],$$

$$R_n = \begin{bmatrix} r_{N-n} \\ r_{N-n+1} \\ r_{N-n+2} \\ \vdots \\ r_N \end{bmatrix}, \quad \Phi_n = \begin{bmatrix} 0 \\ \gamma_{N-n+1} \ (x_{N-n}) \\ \gamma_{N-n+2} \ (x_{N-n}) \\ \vdots \\ \gamma_N \ (x_{N-n}) \end{bmatrix}.$$

If $a(x, y) \leqslant 0$ in domain D, then $\alpha_{n+1} \geqslant 1$ ($x_{n+1} \leqslant x \leqslant a_2$, $n = 0, 1, \ldots, N - 1$). For other values of $a(x, y)$ it is sufficient to take $0 < h < 2/A$, $A = \max\limits_D |a(x, y)|$, for α_{n+1} to be positive.

We will reduce (7) to the form

$$\Gamma_n' = P_n^{-1} Q_n \Gamma_n + P_n^{-1} R_n, \ \Gamma_n(x_{N-n}) = \Phi_n, \quad (8)$$
$$x_{N-n} \leqslant x \leqslant x_{N-n-1}, \ n = 0, 1, \ldots, N-1.$$

Using known results (Ref. 3), we will estimate the solutions of the problems (8) on the basis of the norm (using norm 1):

$$\|\Gamma_n\| \leqslant \|\Phi_n\| \exp \int\limits_{x_{N-n}}^{x} \|P_n^{-1} Q_n\| dv + \int\limits_{x_{N-n}}^{x} \|P_n^{-1} R_n\| \times$$
$$\times \exp \int\limits_{w}^{x} \|P_n^{-1} Q_n\| dv dw, \ x_{N-n} \leqslant x \leqslant x_{N-n-1}, \ n = 0, 1, \ldots, N-1. \quad (9)$$

It is not difficult to find quadratic matrices $P_n^{-1} Q_n$ of the order of $n + 1$ and column matrices $P_n^{-1} R_n$ of the same dimensionality and estimate them from the norm:

$$\|P_n^{-1} Q_n\| \leqslant \xi/\alpha + nh \Delta\lambda_0 \alpha^{-2}, \ \|P_n^{-1} R_n\| \leqslant (n+1) r \lambda_1/\alpha, \quad (10)$$
$$n = 0, 1, \ldots, N-1,$$

where

$$\xi = \max_{x_i < x < a_2} \max_{1 < i < N} |\xi_i| < B + \frac{1}{2} h(C + B_y);$$ /8

$$B = \max_D |b(x, y)|; \ C = \max_D |c(x, y)|;$$

$$B_y = \max_D |b_y(x, y)|; \ \alpha = \min_{x_i < x < a_2} \min_{1 < i < N} |\alpha_i|;$$

$$1+\frac{1}{2}hA \geqslant \alpha \geqslant \begin{cases} 1 \text{ for } a(x,\ y) \leqslant 0 \text{ in D,} \\ 1-\frac{1}{2}\ hA \text{ for other values of a(x, y);} \end{cases}$$

$$\Delta = \max_{x_i<x<a_2} \max_{1<i<N-1} |\xi_i\beta_i + \eta_i\alpha_i|/h;$$

$$\Delta \leqslant AB + C + B_y + \frac{1}{2}hA(C + B_y);$$

$$\lambda_0 = \max_{1<k<l-1} \max_{1<i<N} \lambda^{i-k-1} = \begin{cases} 1 & \text{for } a(x,\ y) \leqslant 0 \text{ in D,} \\ \lambda^{N-2} & \text{for other values of a(x, y);} \end{cases}$$

$$\lambda = \max_{x_i<x<a_2} \max_{1<i<N} |\beta_i/\alpha_i|;$$

$$\lim_{h\to 0} \lambda^{N-2} \leqslant \exp Ad;\ \lambda_1 = \max_{1<i,\ k<N} \lambda^{i-k} =$$

$$= \begin{cases} 1 & \text{for } a(x,\ y) \leqslant 0 \text{ in D,} \\ \lambda^{N-1} & \text{for other values of a(x, y);} \end{cases} \qquad r = \max_{x_i\leqslant x<a_2} \max_{1<i<N} |r_i|;$$

$$r \leqslant h^3 [M_2(C + B_y) + M_2^1 A]/12;\ M_2 = \max_D |u_{yy}|;\ M_2^1 = \max_D |u_{xyy}|.$$

Note that $\|\Phi_0\| = 0$, $\|\Phi_i\| \leqslant \max_{x_{i-1}<x<x_i} \|\Gamma_{i-1}\|$, $i = 1, 2, \ldots,$ N - 1. From (10) and the note just made, we obtain from (9)

$$\|\Gamma_n\| \leqslant R\Big\{\exp[(n+1)\tau + S_n t] - \exp(n\tau + S_n t) +$$
$$+ \frac{2}{1+\delta}\exp(n\tau + S_n t) - \frac{2}{1+\delta}\exp[(n-1)\tau + (S_n - S_1)t] +$$
$$+ \frac{3}{1+2\delta}\exp[(n-1)\tau + (S_n - S_1)t] - \ldots -$$
$$- \frac{n}{1+(n-1)\delta}\exp(\tau + nt) + \frac{n+1}{1+n\delta}\exp(\tau + nt) - \frac{n+1}{1+n\delta}\Big\},$$
$$x_{N-n} \leqslant x \leqslant x_{N-n-1},\ n = 0,\ 1, \ldots,\ N-1,$$

where

$$S_n = \frac{1}{2}n(n+1),\ \tau = h\xi/\alpha p,\ p = \min_{a_1<x<a_2} |L'(x)|;$$
$$t = h^2\Delta\lambda_0/\alpha^2 p,\ \delta = t/\tau,\ R = r\lambda_1/\xi,$$

i.e.,

$$
\begin{gathered}
|\gamma_n(x)| \leqslant \|\Gamma_{N-1}\| \leqslant R\{\exp(N\tau + S_{N-1}t) + \\
+ [2/(1+\delta) - 1]\exp[(N-1)\tau + S_{N-1}t] + \\
+ [3/(1+2\delta) - 2/(1+\delta)]\exp[(N-2)\tau + (S_{N-1} - S_1)t] + \\
+ \ldots + [(1-\delta)/(1-\delta+2k\delta+k^2\delta^2-k\delta^2)] \times \\
\times \exp[(N-k)\tau + (S_{N-1} - S_{k-1})t] +
\end{gathered}
$$

$$
\begin{gathered}
+ \ldots + [N/(1+(N-1)\delta) - (N-1)/(1+(N-2)\delta)] \times \\
\times \exp[\tau + (N-1)t] - [N/(1+(N-1)\delta)]\} \leqslant \\
\leqslant \begin{cases} R[\exp(N\tau + S_{N-1}t) - 1], & \delta \geqslant 1, \\ R[\exp(N\tau + S_{N-1}t) - 1]/[1+(N-1)\delta], & \delta < 1, \end{cases} \\
x_n \leqslant x \leqslant a_2, \; n = 1, 2, \ldots, N. \qquad (11)
\end{gathered}
$$ /9

From (11), replacing R, τ, t, δ by their expressions and taking into account the estimate for r, we obtain the final error estimate:

$$
\gamma \leqslant \begin{cases} \dfrac{h^3[M_2(C+B_y) + M_2^1 A]\lambda_1}{12\xi}\left[\exp\dfrac{2d\xi\alpha + d(d-h)\Delta\lambda_0}{2\alpha^2 p} - 1\right], & h \geqslant \dfrac{\xi\alpha}{\Delta\lambda_0}, \\ \dfrac{h^2[M_2(C+B_y) + M_2^1 A]\lambda_1\alpha d}{12[\xi\alpha + (d-h)\Delta\lambda_0]}\left[\exp\dfrac{2d\xi\alpha + d(d-h)\Delta\lambda_0}{2\alpha^2 p} - 1\right], & h < \dfrac{\xi\alpha}{\Delta\lambda_0}. \end{cases} \qquad (12)
$$

If we consider the equation $u_{xy} = c(x, y)u + f(x, y)$ instead of (1) in the problem (1), (2), then the estimate (12) takes on a much simpler and more convenient form ($\tau = h^2 C/2p$, $\delta = 2$, $R = 2r/hC$):

$$
\gamma \leqslant h^2 M_2[\exp(d^2C/2p) - 1]/6.
$$

Let us note in conclusion that error estimates for the method of integral ratios for equation (1), analogous to those cited above, may be just as conveniently obtained in solving problems whose respective conditions are based on a characteristic and a piecewise smooth curve both originating from the same point.

We will also note that the above exposition may be generalized in its entirety to cover the case where the problem being considered is based on a system of equations of the form

$$\frac{\partial^2 u_i}{\partial x \partial y} = a_i(x, y) \frac{\partial u_i}{\partial x} + \sum_{j=1}^{m} [b_{ij}(x, y) \frac{\partial u_j}{\partial y} + c_{ij}(x, y) u_j] + f_i(x, y), \quad i = 1, 2, \ldots, m,$$

in which the right-hand sides of the equations may even be quasilinear (only the partial derivatives in y are required to be linear).

Belorussian State University Submitted 28 May 1963

References

1. Dorodnitsyn, A. A. Trudy III matematicheskogo s"yezda, 1956 g., 3, 447, Izd. AN SSSR, 1958.
2. Belotserkovskiy, O. M., Chushkiy, P. I. Zhurnal vychislitel'noy matematiki i matematicheskoy fiziki, 2, No. 5, 731, 1962.
3. Lozinskiy, S. M. Izvestiya vysshikh uchebnykh zavedeniy, seriya matematicheskaya, No. 5, 1958.

Translated for the National Aeronautics and Space Administration by John F. Holman and Co. Inc.

"The aeronautical and space activities of the United States shall be conducted so as to contribute . . . to the expansion of human knowledge of phenomena in the atmosphere and space. The Administration shall provide for the widest practicable and appropriate dissemination of information concerning its activities and the results thereof."

—NATIONAL AERONAUTICS AND SPACE ACT OF 1958

NASA SCIENTIFIC AND TECHNICAL PUBLICATIONS

TECHNICAL REPORTS: Scientific and technical information considered important, complete, and a lasting contribution to existing knowledge.

TECHNICAL NOTES: Information less broad in scope but nevertheless of importance as a contribution to existing knowledge.

TECHNICAL MEMORANDUMS: Information receiving limited distribution because of preliminary data, security classification, or other reasons.

CONTRACTOR REPORTS: Technical information generated in connection with a NASA contract or grant and released under NASA auspices.

TECHNICAL TRANSLATIONS: Information published in a foreign language considered to merit NASA distribution in English.

TECHNICAL REPRINTS: Information derived from NASA activities and initially published in the form of journal articles.

SPECIAL PUBLICATIONS: Information derived from or of value to NASA activities but not necessarily reporting the results of individual NASA-programmed scientific efforts. Publications include conference proceedings, monographs, data compilations, handbooks, sourcebooks, and special bibliographies.

Details on the availability of these publications may be obtained from:

SCIENTIFIC AND TECHNICAL INFORMATION DIVISION

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

Washington, D.C. 20546